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A new non-Grassmannian pseudoclassical action for spin-$\frac{1}{2}$ particles

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Abstract. The first-quantised 'spinor chain' path-integral description of spin-$\frac{1}{2}$ particles due to Jacobson is a novel non-Grassmannian approach to the problem of finding a path integral which yields the Dirac or Weyl equation. Unlike conventional path integrals, the spinor chain description does not explicitly use a path integral of the form $\int \exp \mathrm{i}S$, instead summing over all contributing spinor chains (paths) with equal weight, and hence the question of the existence of a canonical pseudoclassical action is left open. We examine the possibility of recasting the spinor chain system in the form of an action, and show that this involves subtleties concerning the weighting of unruly paths. We find the true action corresponding to the spinor chain which when canonically quantised yields the Weyl equation. Our action is extendable to the problem of the massive Dirac equation in a manner unlike the spinor chain. We discuss the classical solutions and compare these new actions with a similar non-Grassmannian action due to Barut and co-workers.

1. Introduction

It is well known that the Dirac equation can be obtained from the canonical quantisation of a pseudoclassical mechanics if the spacetime coordinates of the particle are augmented with Grassmann-valued coordinates. The Dirac equation alone arises if the Grassmann coordinates are spacetime vectors [1], and a supersymmetric spectrum if they are spinors [2]. Quantisation of such pseudoclassical systems has been of interest recently in connection with attempts to covariantly quantise the Green–Schwarz superstring, which is a stringy extension of the superparticle [3].

Aside from twistors [4], Jacobson's spinor chain [5, 6] and the action of Barut *et al* [7] are non-Grassmanian systems which are capable of describing spin-$\frac{1}{2}$ particles. The investigation of alternative types of non-Grassmanian 'internal' coordinates which yield spin-$\frac{1}{2}$ equations is of some interest from the point of view of the extension to an alternative description of strings, but also, as with superspace, might offer an alternative description of the manifold over which quantum field theory is defined.

In this paper we begin with a simple derivation of Jacobson's spinor chain for the case of the Weyl equation (massless neutrino), emphasising its attractive 'pregeometrical' interpretation. We show that the spinor chain path integral, which is

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an extension to 3+1 dimensions of Feynman's checkerboard path integral in 1+1 dimensions [8], may be redescribed formally in the form $\int \exp \mathrm{i}S$, but that this action is 'false' in the sense that it yields the incorrect theory upon canonical quantisation. However, a similar action is found which does yield the Weyl equation in a very straightforward manner. There are two methods of derivation of the path integral form, one of which reduces to the spinor chain. Our action is then generalised to describe massive Dirac particles, where the mass parameter appears in the action in the conventional manner. Our action is closely similar to an action initiated by Proca [9] and further developed by Barut [7] in which, however, mass is introduced as an integral of the motion.

2. The spinor chain form of the Weyl equation

Our treatment of the path integral for the Weyl equation parallels the conventional derivation of Feynman's phase space path integral for non-relativistic particle mechanics [10].

The Weyl equation is

$$-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla\psi(\boldsymbol{x},t)=\mathrm{i}\frac{\partial}{\partial t}\psi(\boldsymbol{x},t). \tag{1}$$

Defining the spinor-valued state vector $|\psi(t)\rangle$ and dividing the time into equal small steps Δt we have

$$|\psi(t+\Delta t)\rangle=\exp\{-\mathrm{i}\Delta t\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\mathrm{op}}\}|\psi(t)\rangle \tag{2}$$

where $\boldsymbol{p}_{\mathrm{op}}=-\mathrm{i}\nabla$. The classical trajectories are described by $(\boldsymbol{x}(t), s(t))$, where $s(t)=\pm 1$ is an eigenvalue of σ_3. We write the corresponding eigenvector as $|\boldsymbol{x}(t)\rangle\alpha(t)$ where $\alpha(t)$ is the eigenvector of σ_3 with eigenvalue $s(t)$. The Feynman kernel between times t and $t+\Delta t$ is thus written as

$$K[\boldsymbol{x}(t),s(t);\boldsymbol{x}(t+\Delta t),s(t+\Delta t)]=\alpha^{\dagger}(t+\Delta t)\langle\boldsymbol{x}(t+\Delta t)|\exp\{-\mathrm{i}\Delta t\boldsymbol{\sigma}\cdot\boldsymbol{p}_{\mathrm{op}}\}|\boldsymbol{x}(t)\rangle\alpha(t). \tag{3}$$

Defining the spin matrix-valued kernel $\bar{K}$ by

$$K[\boldsymbol{x}(t),s(t);\boldsymbol{x}(t'),s(t')]=\alpha^{\dagger}(t')\bar{K}[\boldsymbol{x}(t);\boldsymbol{x}(t')]\alpha(t) \tag{4}$$

inserting the unit operator

$$1=\int \mathrm{d}^3p|\boldsymbol{p}\rangle\langle\boldsymbol{p}|=\int \mathrm{d}^3x|\boldsymbol{x}\rangle\langle\boldsymbol{x}|$$

and using the relation

$$\langle\boldsymbol{x}|\boldsymbol{p}\rangle=\frac{1}{(\sqrt{2\pi})^3}\exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x})$$

we obtain in the usual fashion

$$\bar{K}[\boldsymbol{x}(t);\boldsymbol{x}(t+\Delta t)]=\frac{1}{(2\pi)^3}\int \mathrm{d}^3p(t)\exp\{\mathrm{i}\boldsymbol{p}(t)\cdot(\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t))\}\exp\{-\mathrm{i}\Delta t\boldsymbol{\sigma}\cdot\boldsymbol{p}(t)\}. \tag{5}$$

Between arbitrary times t_i and $t_f \equiv t_i + N\Delta t$ the kernel is

$$\begin{aligned} \bar{K}[\boldsymbol{x}(t_i); \boldsymbol{x}(t_f)] = \left[\frac{1}{(2\pi)^3}\right]^N \int \mathrm{d}^3p(t_f-\Delta t)\ldots \mathrm{d}^3p(t_i) \int \mathrm{d}^3x(t_f-\Delta t)\ldots \mathrm{d}^3x(t_i+\Delta t) \\ \times \exp\left\{\mathrm{i} \sum_{t=t_i}^{t_f-\Delta t} \boldsymbol{p}(t)\cdot[\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)]\right\} \\ \times \exp\{-\mathrm{i}\Delta t\boldsymbol{p}(t_f-\Delta t)\cdot\boldsymbol{\sigma}\}\ldots\exp\{-\mathrm{i}\Delta t\boldsymbol{p}(t_i)\cdot\boldsymbol{\sigma}\}. \end{aligned} \tag{6}$$

The derivation of the spinor chain form, following Jacobson [5], consists of introducing a complex 2-component spinor z and writing, for Δt small,

$$\begin{aligned} \exp\{-\mathrm{i}\Delta t\boldsymbol{p}(t)\cdot\boldsymbol{\sigma}\} &= 1_2 - \mathrm{i}\Delta t\boldsymbol{p}(t)\cdot\boldsymbol{\sigma} \\ &= \int \mathrm{d}^2z(t)z(t)z(t)^\dagger[1-3\mathrm{i}\Delta\boldsymbol{p}(t)\cdot z(t)^\dagger\boldsymbol{\sigma}z(t)] \\ &= \int \mathrm{d}^2z(t)z(t)z(t)^\dagger \exp\{-\mathrm{i}3\Delta t\boldsymbol{p}(t)\cdot z(t)^\dagger\boldsymbol{\sigma}z(t)\} \end{aligned} \tag{7}$$

using

$$\int \mathrm{d}^2zzz^\dagger(z^\dagger\boldsymbol{\sigma}z) = \tfrac{1}{3}\boldsymbol{\sigma} \qquad \int \mathrm{d}^2zzz^\dagger = 1_2 \qquad \int \mathrm{d}^2z = 2 \tag{8}$$

where

$$z \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

satisfies $z^\dagger z = 1$ and the measure d^2z is appropriately normalised (or, alternatively, z is an element of CP^1, since the phase contributes only to normalisation). Equation (6) becomes

$$\begin{aligned} \bar{K}[\boldsymbol{x}(t_i); \boldsymbol{x}(t_f)] = \left[\frac{1}{(2\pi)^3}\right]^N \int \mathrm{d}^3p(t_f-\Delta t)\ldots \mathrm{d}^3p(t_i) \\ \times \int \mathrm{d}^3x(t_f-\Delta t)\ldots \mathrm{d}^3x(t_i+\Delta t) \\ \times \int \mathrm{d}^2z(t_i)\ldots \mathrm{d}^2z(t_f-\Delta t)z(t_f-\Delta t)z^\dagger(t_f-\Delta t)\ldots z(t_i)z^\dagger(t_i) \\ \times \exp\left\{\mathrm{i}\sum_{t=t_i}^{t_f-\Delta t} \boldsymbol{p}(t)\cdot[\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)-3z^\dagger(t)\boldsymbol{\sigma}z(t)\Delta t]\right\}. \end{aligned} \tag{9}$$

Since $z^\dagger\boldsymbol{\sigma}z$ is real, each p-integration is just a delta-function:

$$\begin{aligned} \frac{1}{(2\pi)^3}\int \mathrm{d}^3p(t)\exp\{\mathrm{i}\boldsymbol{p}(t)\cdot[\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)-3z^\dagger(t)\boldsymbol{\sigma}z(t)\Delta t]\} \\ = \delta^{(3)}(\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t)-3z^\dagger(t)\boldsymbol{\sigma}z(t)\Delta t) \end{aligned} \tag{10}$$

and hence equation (9) can be written symbolically in the limit $\Delta t \to 0$ using the delta functional $\delta[\ ]$ as

$$\bar{K} = \int \mathrm{D}z(t)\prod_t[z(t)z^\dagger(t)]\int \mathrm{D}\boldsymbol{x}(t)\delta[\mathrm{d}\boldsymbol{x}(t)-3z^\dagger(t)\boldsymbol{\sigma}z(t)\,\mathrm{d}t]. \tag{11}$$

The x-integrations merely serve to 'pregeometrically' define each successive path element $\mathrm{d}\boldsymbol{x}(t)$ in terms of the unit spinor $z(t)$ and the kernel can be recast in the form given by Jacobson [5]:

$$\bar{K} = \int \mathrm{d}^3 z(t_i) \ldots \mathrm{d}^2 z(t_f - \Delta t) z(t_f - \Delta t) z^\dagger(t_f - \Delta t) \ldots z(t_i) z^\dagger(t_i)$$

$$\times \delta^{(3)}\left(\boldsymbol{x}_f - \boldsymbol{x}_i - 3 \sum_{t=t_i}^{t_f - \Delta t} z^\dagger(t) \boldsymbol{\sigma} z(t) \Delta t\right) \tag{12}$$

where only the path endpoints appear explicitly rather than the full paths.

In equation (9), the z-integrations match the p-integrations, which as usual number one more than the x-integrations. In the definition we chose $\boldsymbol{p}(t)$ and $z(t)$ to vary at t_i and to be fixed at t_f, but a more symmetrical possibility is to define z and $\boldsymbol{p}$ at the mid-points between the t_k. We can then view the initial and final states $\alpha(t_i)$ and $\alpha(t_f)$ as $z(t_i)$ and $z(t_f)$ and write symbolically

$$K[\boldsymbol{x}(t_i), z(t_i); \boldsymbol{x}(t_f), z(t_f)] \equiv z^\dagger(t_f) \bar{K}[\boldsymbol{x}(t_i); \boldsymbol{x}(t_f)] z(t_i)$$

$$= \int \mathrm{D}z(t) \mathrm{D}\boldsymbol{p}(t) \mathrm{D}\boldsymbol{x}(t) \exp\{\mathrm{i}S\} \tag{13}$$

where

$$S = \int \mathrm{d}t [\tfrac{1}{2} i (z^\dagger \dot{z} - \dot{z}^\dagger z) + \boldsymbol{p} \cdot (\dot{\boldsymbol{x}} - 3 z^\dagger \boldsymbol{\sigma} z)] \tag{14}$$

and we have used

$$\begin{aligned} z^\dagger(t + \Delta t) z(t) &= 1 + \tfrac{1}{2}\{[z^\dagger(t + \Delta t) - z^\dagger(t)] z(t) - z^\dagger(t + \Delta t)[z(t + \Delta t) - z(t)]\} \\ &\approx \exp\{-\tfrac{1}{2}(z^\dagger \dot{z} - \dot{z}^\dagger z) \Delta t\} \end{aligned} \tag{15}$$

which is true only for continuous $z(t)$ as $\Delta t \to 0$. That the formal expression (13) has been achieved does not necessarily mean that (14) is the canonical action, by which we mean the action that yields the Weyl equation upon canonical quantisation. The fact that the integrand in (13) is valid only for continuous $z(t)$ is not in itself a concern: this is a common occurrence in path integrals derived from canonical actions, a relevant example of which is Klauder's non-Grassmannian representation of a Fermi oscillator [11], and it occurs even for the ordinary harmonic oscillator [11]. The decision as to whether or not (14) represents the true canonical action for a neutrino rests on whether or not the unambiguously defined expression (9) specifies the same weighting for discontinuous trajectories as results from the *conventional derivation* of the path integral (involving the insertion of resolutions of unity) from the Hamiltonian of (14). In what follows we will show that this is not the case, and that in fact the canonical action does not contain a factor of 3.

3. The Weyl equation via canonical quantisation

First recall that an action which is the canonical form

$$S = \int \mathrm{d}t [\pi \dot{q} - H(\pi, q)]$$

with canonical coordinate q and momentum π may be modified with the addition of a boundary term to

$$S=\int \mathrm{d}t[\tfrac{1}{2}(\pi\dot{q}-q\dot{\pi})-H(\pi, q)]$$

and re-expressed in terms of the complex variable

$$z=\frac{1}{\sqrt{2}}(q+\mathrm{i}\pi)$$

as

$$S=\int \mathrm{d}t[\tfrac{1}{2}\mathrm{i}(z^*\dot{z}-\dot{z}^*z)-H(z, z^*)]$$

yielding the Poisson bracket relations

$$[z, z^*]_{\mathrm{PB}}=-\mathrm{i}$$

which become the familiar harmonic oscillator raising and lowering operators in the quantum theory, satisfying

$$[z, z^*]=1.$$

Taking the obvious many-variable extension, it is clear that the action given by

$$S_I=\int \mathrm{d}t\left[\frac{\mathrm{i}}{2}(z^\dagger\dot{z}-\dot{z}^\dagger z)+\boldsymbol{p}\cdot\dot{\boldsymbol{x}}-\boldsymbol{p}\cdot z^\dagger\boldsymbol{\sigma}z\right] \tag{16}$$

is already in canonical form. We define the two-component spinor z to satisfy $z^\dagger z=1$ as in the spinor chain formulation. In the language of Dirac [14] this is a first-class constraint, in contrast to the Grassmannian description of the Dirac equation [1] which requires passing through the Lagrangian formalism, computing second-class phase space constraints and defining the Dirac brackets. The Schrödinger equation following from (16) is thus straightforwardly given by

$$H|\psi\rangle=\boldsymbol{p}_{\mathrm{op}}\cdot z^\dagger_{\mathrm{op}}\boldsymbol{\sigma}_{\mathrm{op}}|\psi\rangle=\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle \tag{17}$$

and the unimodularity constraint becomes a weak condition on the physical states

$$z^\dagger_{\mathrm{op}}z_{\mathrm{op}}|\psi\rangle=|\psi\rangle \tag{18}$$

which restricts the unbounded spectrum of states of the form

$$|n_1\rangle\otimes|n_2\rangle$$

where $|n_i\rangle$, $n_i\geqslant 0$, is the harmonic oscillator level given by

$$z^\dagger_{(i)\mathrm{op}}z_{(i)\mathrm{op}}|n_i\rangle=n_i|n_i\rangle$$

to the form

$$\langle\boldsymbol{x}|\psi\rangle=\psi_1(\boldsymbol{x}, t)|1_1\rangle\otimes|0_2\rangle+\psi_2(\boldsymbol{x}, t)|0_1\rangle\otimes|1_2\rangle. \tag{19}$$

This yields

$$z^\dagger_{\mathrm{op}}\boldsymbol{\sigma}z_{\mathrm{op}}|\psi\rangle=\boldsymbol{\sigma}|\psi\rangle \tag{20}$$

which can be verified by explicit calculation. The role of $z_{\rm op}$ in Hermitian bilinear forms is essentially mute. hence

$$\psi(\boldsymbol{x}, t) \equiv \begin{bmatrix} \psi_1(\boldsymbol{x}, t) \\ \psi_2(\boldsymbol{x}, t) \end{bmatrix}$$

obeys the Weyl equation (1).

4. Derivation of the path integral

In order to construct the phase space path integral we use the normalised states

$$|z\rangle \equiv z_{(1)}|1_1\rangle \otimes |0_2\rangle + z_{(2)}|0_1\rangle \otimes |1_2\rangle \tag{21}$$

where $z^\dagger z = 1$. These are analogous to coherent states and satisfy

$$\langle z|z'\rangle = z^\dagger z' \qquad \langle z|z_{\rm op}^\dagger \boldsymbol{\sigma} z_{\rm op}|z'\rangle = z^\dagger \boldsymbol{\sigma} z'. \tag{22}$$

(These are unitarily equivalent to the 'coherent spin states' for spin $\frac{1}{2}$ [12].) The above identities mean that the resolution of unity given by

$$1 = \int \mathrm{d}^2 z |z\rangle\langle z| \tag{23}$$

plays the same role in the path integral as

$$1 = \int \mathrm{d}^2 z\, z z^\dagger \tag{24}$$

mentioned earlier. The conventional derivation of the path integral consists in inserting the various resolutions of unity (such as (23)) between the exponential factors in the expression

$$K = \langle z(t_f), \boldsymbol{x}(t_f)| \exp\{-\mathrm{i}\Delta t H_{\rm op}\} \exp\{-\mathrm{i}\Delta t H_{\rm op}\} \ldots \exp\{-\mathrm{i}\Delta t H_{\rm op}\}|z(t_i), \boldsymbol{x}(t_i)\rangle. \tag{25}$$

After using

$$\langle z'|\exp\{-\mathrm{i}\Delta t H_{\rm op}\}|z\rangle \approx \langle z'|z\rangle\left(1 - \mathrm{i}\Delta t \frac{\langle z'|H_{\rm op}|z\rangle}{\langle z'|z\rangle}\right)$$

for small Δt this yields

$$\begin{aligned} K = \left[\frac{1}{(2\pi)^3}\right]^N & \int \mathrm{d}^3 p(t_i) \ldots \mathrm{d}^3 p(t_j - \Delta t) \int \mathrm{d}^3 x(t_i + \Delta t) \ldots \mathrm{d}^3 x(t_f - \Delta t) \\ & \times \int \mathrm{d}^2 z(t_f - \Delta t) \ldots \mathrm{d}^2 z(t_i + \Delta t) z^\dagger(t_f) z(t_f - \Delta t) \\ & \times z^\dagger(t_f - \Delta t) \ldots z(t_i + \Delta t) z^\dagger(t_i + \Delta t) z(t_i) \\ & \times \exp\left\{\mathrm{i} \sum_{t=t_i}^{t_f - \Delta t} \boldsymbol{p}(t) \cdot \left[\boldsymbol{x}(t + \Delta t) - \boldsymbol{x}(t) - \frac{z^\dagger(t+\Delta t)\boldsymbol{\sigma} z(t)}{z^\dagger(t+\Delta t) z(t)} \Delta t\right]\right\}. \end{aligned} \tag{26}$$

As expected, this may be written symbolically as a path integral over $\exp\{\mathrm{i}S_{\rm I}\}$, with validity only for continuous $z(t)$ trajectories, i.e. when $z(t+\Delta t) \to z(t)$ as $\Delta t \to 0$. Clearly if we had considered equation (14) as the action, the equivalent of (26) would specify a different weighting for the discontinuous $z(t)$ trajectories than that of the spinor chain (equation (9)). The spinor chain corresponds to an alternative path integral derivation which is always possible if the Hamiltonian can be expressed in a form diagonal with respect to the coherent states [13]:

$$H_{\mathrm{op}} = \int \mathrm{d}^2 z\, h(z)|z\rangle\langle z|. \tag{27}$$

Using the identities (8) and (22)-(24) we find

$$h(z) = 3\boldsymbol{p} \cdot (z^{\dagger}\boldsymbol{\sigma} z). \tag{28}$$

Rather than inserting between the exponentials in (25), when (27) holds an alternative is to convert each exponential directly since (at least in this case)

$$\exp\{-\mathrm{i}\Delta t H_{\mathrm{op}}\} = \int \mathrm{d}^2 z\, \exp\{-\mathrm{i}\Delta t h(z)\}|z\rangle\langle z|. \tag{29}$$

Thus the factor of 3 in the spinor chain is seen to correspond to an alternative derivation of the path integral from the canonical action (16), one which does not guarantee to produce the path integrand in the form of an exponential of the canonical action.

In retrospect it is of interest to note that Klauder [15] suggested long ago the exercise of starting with the Weyl Hamiltonian and searching for the corresponding pseudoclassical theory, making use of over-complete states including unimodular spinors for the spin degrees of freedom.

5. Pseudoclassical solutions and extension to the massive Dirac equation

It is instructive to examine the pseudoclassical equations of motion resulting from S_{I}. We use the word pseudoclassical to emphasise that these equations of motion, as in the more familiar Grassmannian formulations, are not necessarily the classical limit ($h \to 0$) of the quantum system. This point will be clarified later. The pseudoclassical Hamilton equations are

$$\dot{z} = -\mathrm{i}\boldsymbol{\sigma} \cdot \boldsymbol{p} z \qquad \dot{\boldsymbol{p}} = 0 \qquad \dot{\boldsymbol{x}} = z^{\dagger}\boldsymbol{\sigma} z \tag{30}$$

with solution

$$z(t) = \exp\{-\mathrm{i}\boldsymbol{\sigma} \cdot \boldsymbol{p} t\} z(0) \qquad \boldsymbol{p}(t) = \boldsymbol{p} \tag{31}$$

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \frac{1}{2p}\sin(2pt)\dot{\boldsymbol{x}}(0) + \frac{1}{2p}(1 - \cos(2pt))\hat{\boldsymbol{p}} \times \dot{\boldsymbol{x}}(0) + \left(t - \frac{1}{2p}\sin(2pt)\right)\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}} \cdot \dot{\boldsymbol{x}}(0)) \tag{32}$$

where $\hat{\boldsymbol{p}}$ is a unit vector in the direction of $\boldsymbol{p}$. The instantaneous speed is always c and the particle in general moves in a corkscrew fashion at an average speed less than c. When the velocity and momentum are aligned, i.e.

$$\hat{\boldsymbol{p}} \cdot \dot{\boldsymbol{x}} = \pm 1 \tag{33}$$

the particle undergoes rectilinear motion on the light cone. Oddly, the *quantum* equation for the energy-momentum eigenstates $|\phi\rangle$, namely

$$\boldsymbol{p} \cdot \boldsymbol{\sigma}|\phi\rangle = E|\phi\rangle \Rightarrow E = \pm p \tag{34}$$

ensures that (33) holds, restricting physical neutrinos to the light cone, since pseudoclassically

$$E = p(\hat{\boldsymbol{p}} \cdot \dot{\boldsymbol{x}}). \tag{35}$$

The negative sign corresponds to the negative energy neutrino solution. Thus in this model the observed rectilinear light-cone motion of neutrinos is a quantum effect, caused by the quantisation of $\hat{\boldsymbol{p}} \cdot \dot{\boldsymbol{x}}$. Note that if we reinsert the factors of $\hbar$ into the pseudoclassical action there is a factor of $\hbar$ in front of the kinetic terms of $z(t)$ and the pseudoclassical spin, for instance, is given by

$$\boldsymbol{S} = \tfrac{1}{2}\hbar z^{\dagger}\boldsymbol{\sigma}z.$$

Such factors of $\hbar$ are also present in the Grassmannian pseudoclassical actions.

The straightforward extension of S_{I} to the case of the massive Dirac equation is (hereafter in this section explicitly showing $\hbar$ factors)

$$S_{\mathrm{II}} = \int \mathrm{d}t \left[\frac{\mathrm{i}\hbar}{2}(z^{\dagger}\dot{z} - \dot{z}^{\dagger}z) + \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - \boldsymbol{p}\cdot z^{\dagger}\boldsymbol{\alpha}z - mz^{\dagger}\beta z\right] \tag{36}$$

where z is a four-component spinor obeying the unimodularity condition $z^{\dagger}z = 1$, and $\boldsymbol{\alpha}$ and β are the Dirac matrices.

Repetition of the arguments for the Weyl case yields the massive Dirac equation. The pseudoclassical Hamilton equations are

$$\dot{z} = -\frac{\mathrm{i}}{\hbar}(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)z \qquad \dot{\boldsymbol{p}} = 0 \qquad \dot{\boldsymbol{x}} = z^{\dagger}\boldsymbol{\alpha}z \tag{37}$$

with solution

$$\begin{aligned} z(t) &= \exp\left\{-\frac{\mathrm{i}}{\hbar}(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)t\right\}z(0) \\ \boldsymbol{x}(t) &= \boldsymbol{x}(0) + \frac{\hbar}{2\varepsilon}\sin(2\varepsilon t)z^{\dagger}(0)\boldsymbol{\alpha}z(0) \\ &\quad + \frac{\hbar}{2\varepsilon}(1 - \cos(2\varepsilon t))\left[\frac{\boldsymbol{p}}{\varepsilon}\times z^{\dagger}(0)\boldsymbol{\Sigma}z(0) + \mathrm{i}\frac{m}{\varepsilon}z^{\dagger}(0)\beta\boldsymbol{\alpha}z(0)\right] \\ &\quad + \left[t - \frac{\hbar}{2\varepsilon}\sin(2\varepsilon t)\right]\boldsymbol{p}\frac{E}{\varepsilon^2} \end{aligned} \tag{38}$$

where

$$\varepsilon \equiv +\sqrt{|\boldsymbol{p}^2| + m^2} \qquad \text{and} \qquad \boldsymbol{\Sigma} \equiv \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix}$$

and the energy E is given by (for all times)

$$E = \boldsymbol{p}\cdot z^{\dagger}(0)\boldsymbol{\alpha}z(0) + mz^{\dagger}(0)\beta z(0). \tag{39}$$

In analogy with the Weyl case, the *quantum* equation for the momentum eigenstates

$$(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)|\phi\rangle = E|\phi\rangle \tag{40}$$

quantises the energy so that $E = \pm\varepsilon$. This condition therefore imposes in turn a condition on $z(0)$, which is that $z(0)$ also obey (40). Inserting the solutions of (40) in (38) yields, as for the Weyl case,

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) \pm t\frac{\boldsymbol{p}}{\varepsilon}. \tag{41}$$

This is the class of pseudoclassical trajectories which obey the quantum energy-momentum relationship. As a consistency check it can be verified that the familiar operator solution for $\boldsymbol{x}_{\mathrm{op}}(t)$ [16] is obtained (after some manipulation) by inserting operator labels into (38). That the *Zitterbewegung* terms disappear in (41) is due to the fact that the relevant terms in the operator solution have non-vanishing matrix elements only between states of opposite energies, and the classical trajectory (41) is identified by the correspondence

$$\boldsymbol{x}(t) = \langle\phi|\boldsymbol{x}_{\mathrm{op}}(t)|\phi\rangle.$$

The generalisation of (28) to unimodular 4-spinors is

$$h(z) = 5[\boldsymbol{p}\cdot z^{\dagger}\boldsymbol{\alpha} z + mz^{\dagger}\beta z]$$

which could be used in a diagonal path integral representation. Note, however, that Jacobson's extension to the massive Dirac case was achieved in a different manner, by the use of two unimodular Weyl spinors.

6. A covariant formulation

Although S_{I} and S_{II} lead to Lorentz-invariant quantum equations, they are not themselves Lorentz invariant. This feature is also present in the spinor chain of equation (11) on which they are modelled. Nevertheless a Lorentz-*co*variant expression of the pseudoclassical actions S_{I} and S_{II} can be found since any expression, Lorentz invariant or not, can be written in a *co*variant form with the use of non-dynamical objects. (In the context of theories of gravity, Cartan wrote down Newtonian gravity in a generally covariant way to illustrate that the 'principle of general covariance' by itself has no physical content [17]†.) Such a covariant expression can be found by first writing the unimodularity constraint into the action in the form of a Lagrange multiplier p_0, changing (36) to

$$S = \int \mathrm{d}t\left[\frac{\mathrm{i}}{2}\left(\bar{z}\gamma^0\frac{\mathrm{d}z}{\mathrm{d}t} - \frac{\mathrm{d}\bar{z}}{\mathrm{d}t}\gamma^0 z\right) + \boldsymbol{p}\cdot(\dot{\boldsymbol{x}} - \bar{z}\boldsymbol{\gamma} z) - p_0(1 - \bar{z}\gamma^0 z) - m\bar{z}z\right] \tag{42}$$

where use has been made of the Dirac adjoint $\bar{z} \equiv z^{\dagger}\beta$ and the matrices $\gamma^0 \equiv \beta$, $\boldsymbol{\gamma} \equiv \beta\boldsymbol{\alpha}$. Rewriting (42) as an integral over an arbitrary worldline parameter s,

$$S = \int \mathrm{d}s\left[\frac{\mathrm{i}}{2}\left(\bar{z}\gamma^0\frac{\mathrm{d}z}{\mathrm{d}s} - \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}\gamma^0 z\right) - p_\mu\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \frac{\mathrm{d}t}{\mathrm{d}s}\bar{z}\gamma^\mu z\right) - m\frac{\mathrm{d}t}{\mathrm{d}s}\bar{z}z\right]. \tag{43}$$

Under a Lorentz transformation,

$$\frac{\mathrm{d}t}{\mathrm{d}s} \qquad \text{and} \qquad \bar{a}\gamma^0 b$$

† We are grateful to one of the referees for drawing attention to this point.

transform as

$$n_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \qquad \text{and} \qquad n_\mu \bar{a}\gamma^\mu b$$

respectively, where n_μ is a timelike vector satisfying $n^2=1$. A covariant and reparametrisation invariant expression is therefore

$$S_{\mathrm{III}} = \int \mathrm{d}s \left[\frac{\mathrm{i}}{2}\left(\bar{z} n_\nu \gamma^\nu \frac{\mathrm{d}z}{\mathrm{d}s} - \frac{\mathrm{d}\bar{z}}{\mathrm{d}s} n_\nu \gamma^\nu z \right) - p_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s} - n_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \bar{z}\gamma^\mu z \right) - m n_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \bar{z} z \right] \tag{44}$$

where n_μ is a non-dynamical constant 4-vector with unit magnitude, the unit magnitude replacing the unimodularity condition on z of S_{II}. n_μ encapsulates the Lorentz non-invariance of the pseudoclassical action and points back to the choice of time coordinate involved in the Hamiltonian quantum treatment. By virtue of the Lorentz invariance of the quantum theory one expects the vector n_μ to be arbitrary.

7. Comparison with other non-Grassmannian approaches

Our action has a close similarity to the action of Proca [9] and Barut *et al* [7] which in flat space reads

$$S_{\mathrm{IV}} = \int \mathrm{d}\tau \left[\frac{\mathrm{i}}{2}\left(\bar{z}\frac{\mathrm{d}z}{\mathrm{d}\tau} - \frac{\mathrm{d}\bar{z}}{\mathrm{d}\tau} z \right) + p_\mu \left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} - \bar{z}\gamma^\mu z \right) \right] \tag{45}$$

where z is a complex Dirac spinor unrestricted by a unimodularity condition. Via the Lagrange multiplier role of p_μ, the above action shares with our action and Jacobson's spinor chain the 'pregeometrical' definition of velocity in terms of a spinor bilinear form. However the parameter τ is apparently physically meaningful (not an arbitrary worldline parameter) in the above model, since (45) is not reparametrisation invariant. Mass does not appear as a parameter, being introduced rather as the value of the Hamiltonian with respect to τ, which therefore becomes the proper time of the centre of mass of the particle. The exact relationship between our action and (45) is not yet clear to us.

It is also interesting to note the comparison between the definition inherent in our model of the Weyl equation (and similarly in the above models) of

$$\mathrm{d}\boldsymbol{x} \propto z^\dagger \boldsymbol{\sigma} z \, \mathrm{d}t$$

and the connection between spinors and spacetime advocated by Penrose [4], which uses a connection between the position vector x^μ and a spinor ω of the form

$$x^\mu \propto \omega^\dagger \sigma^\mu \omega.$$

8. Conclusion

The relativistic single-particle spin-$\frac{1}{2}$ systems we have discussed fall within a class of non-Grassmannian approaches which imply a definition of small spacetime displacements in terms of a fundamental spinor variable. The Weyl particle version of our action is the canonical equivalent of Jacobson's spinor chain. In contrast to previous non-Grassmannian work, our approach to the massive Dirac equation is conventional in that mass appears as a parameter in a canonical action. Since in our action there is only one phase space constraint (the unimodularity constraint) which is of the first class, quantisation is simpler than for Grassmannian actions. We have presented a covariant and reparametrisation invariant version which might have a useful extension to fermionic strings.

One unusual feature of our model is that the relationship between energy and momentum $E^2=|\boldsymbol{p}|^2+m^2$ emerges only after quantisation, and hence the pseudo-classical dynamics *per se* does not describe the physically realised classical limit of the quantum theory.

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